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Path integrals, stochastic differential equations and operator ordering in supersymmetric quantum mechanics

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Abstract. We give an explicit relation between the path integral and the operator ordering of the Hamiltonian for supersymmetric quantum mechanics. Using a stochastic differential equation we are able to resolve the ambiguities associated with defining the supersymmetric path integral.

1. Introduction

Supersymmetric quantum mechanics (SUSYQM) was introduced by Witten [1] as a laboratory for understanding supersymmetry breaking in supersymmetric quantum field theories. In further study of SUSYQM [2], Witten showed that upon quantisation, the supersymmetric nonlinear σ-model was equivalent to the Laplacian (De Rham-Kodaira Laplacian) acting on differential forms. In [3] a geometric interpretation of SUSYQM was used to give a new proof of the Morse theorem. Alvarez-Gaumé, Fridan and Windey [4] were able to use geometric interpretations of SUSYQM to derive the index theorem in a manner accessible to physicists by the formal manipulation of the supersymmetric path-integral.

For quantum mechanics on a curved manifold it has been known for a long time that there are ambiguities associated with a careful discretisation treatment of the path integral [5]. When the discretised path integral is compared with the formal path integral, there are extra terms, called extra potential terms. Gervais and Jevicki [6] showed that even on a flat manifold when a point-canonical transformation of the path integral is performed, these extra potential terms arise. We would like to emphasise that we do not question the validity of formal path integral methods for SUSYQM. However, it should be interesting to see if a careful discretisation treatment of the path integral for SUSYQM also leads to additional potential terms when compared with the formal treatment.

The paper has four main sections and an appendix. In section 2 we review the geometrical interpretation of SUSYQM which provides us with a well defined Hamiltonian. In section 3, we use the Nicolai map [7] to perform a point-canonical transformation of the path integral from Cartesian coordinates to curvilinear coordinates on a flat manifold. We find that the existence of a Nicolai map removes the ambiguity associated with defining the path integral, and that the supersymmetric path integral contains no additional potential terms. In section 4 we show that it is possible to extend the Nicolai map to curved manifolds. In the appendix we give our conventions for the fermionic integrals used in the paper.

2. Supersymmetric quantum mechanics

In flat space using Cartesian coordinates, the supersymmetric Hamiltonian is given by the conjugated supercharges Q_c and Q_c^*

$$H = \frac{1}{\hbar}\{Q_c^*, Q_c\} \tag{2.1}$$

where the conjugated supercharges are given by

$$Q_c^* = \frac{i}{\sqrt{2}}\psi^{*a}[P_a - i(\partial_a V)] \tag{2.2}$$

$$Q_c = -\frac{i}{\sqrt{2}}\eta^{ed}\psi_e[P_d + i(\partial_d V)] \tag{2.3}$$

and the operators ψ^{*a}, ψ_a are respectively fermionic creation and annihilation operators which satisfy the anti-commutator relation $\{\psi^{*a}, \psi_b\} = \hbar\delta^a_b$. P_a is the momentum operator $-i\hbar\partial_a$.

The supercharges Q_c^* and Q_c can be identified with the conjugated exterior derivative d_c and its conjugated adjoint δ_c. That is, $Q_c^* = (\hbar/\sqrt{2})d_c = (\hbar/\sqrt{2})\,e^{-U} d\, e^{U}$ and $Q_c = (\hbar/\sqrt{2})\delta_c = (\hbar/\sqrt{2})\,e^{U}\delta\, e^{-U}$, where $V = \hbar U$. The Hamiltonian $H = (\hbar^2/2)(d_c + \delta_c)^2$, is the conjugated Laplacian acting on differential forms, where we now identify p-forms with p-fermion supersymmetric states.

The Hamiltonian (Laplacian) is well-defined, and for a curved Riemannian manifold we have

$$Q_c^* = \frac{i}{\sqrt{2}}\psi^{*\alpha}[\tau_\alpha - i(\nabla_\alpha V)] \tag{2.4}$$

$$Q_c = -\frac{i}{\sqrt{2}}g^{\gamma\delta}\psi_\gamma[\tau_\delta + i(\nabla_\delta V)] \tag{2.5}$$

where $\tau_\mu = P_\mu + i\Gamma^\alpha_{\mu\beta}\psi^{*\beta}\psi_\alpha$ is the covariant derivative $-i\hbar\bar{\nabla}_\mu$, and satisfies the commutator relations

$$\begin{aligned}
&[P_\alpha, x^\beta] = -i\hbar\delta^\beta_\alpha \\
&[P_\alpha, \psi^{*\beta}] = 0 \\
&[P_\alpha, \psi_\beta] = 0 \\
&[\tau_\alpha, x^\beta] = -i\hbar\delta^\beta_\alpha \\
&[\tau_\alpha, \psi^{*\beta}] = i\hbar\Gamma^\beta_{\alpha\mu}\psi^{*\mu} \\
&[\tau_\alpha, \psi_\beta] = -i\hbar\Gamma^\mu_{\alpha\beta}\psi_\mu \\
&[\tau_\alpha, \tau_\beta] = -i\hbar R^\mu{}_{\delta\alpha\beta}\psi^{*\delta}\psi_\mu .
\end{aligned} \tag{2.6}$$

In curved space we have the operator $Q^* = (\hbar/\sqrt{2})\psi^{*\mu}\bar{\nabla}_\mu$. Naively we would expect its adjoint to be $-(\hbar/\sqrt{2})\bar{\nabla}_\mu\psi^\mu$. However, since the scalar product is defined by $(A, B) = \int dx^N g^{1/2}A_{\mu\nu\ldots\rho}B^{\mu\nu\ldots\rho}$, Q^* has adjoint $Q = -(\hbar/\sqrt{2})g^{-1/2}(\bar{\nabla}_\mu g^{1/2}\psi^\mu)$ which is equal to $Q = -(\hbar/\sqrt{2})g^{\mu\nu}\psi_\mu\bar{\nabla}_\nu$ because $g^{-1/2}\bar{\nabla}_\mu g^{1/2} = \Gamma^\alpha_{\mu\alpha}$.

The Hamiltonian $(1/\hbar)\{Q_c^*, Q_c\}$ has a well defined operator ordering,

$$\begin{aligned}
H = {}&\tfrac{1}{2}g^{-1/2}\tau_\mu g^{1/2}g^{\mu\nu}\tau_\nu + \tfrac{1}{2}R_\alpha{}^\beta{}_\gamma{}^\delta\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta \\
&+ \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V) - \tfrac{1}{2}g^{\mu\tau}(\nabla_\mu\nabla_\nu V)[\psi^{*\nu}, \psi_\tau].
\end{aligned} \tag{2.7}$$

On a flat manifold the curvature tensor is zero, and hence the second term on the right-hand side of equation (2.7) will be zero.

3. The path integral

For supersymmetric field theories in flat space there exist Nicolai maps. We shall use the following definition of a Nicolai map [6]. The Nicolai map is a transformation of the bosonic field configurations $\varphi_i(x)$, $N:\varphi_i(x)\to\xi_i(x)$, such that

(*a*) the bosonic part of the Lagrangian is given by

$$L_B=\frac{1}{2}\sum_i \xi_i^2+\text{total divergence}$$

(*b*) the Jacobian of the transformation $\det(\delta\xi_i/\delta\varphi_i)$ is given by the determinant for the fermions

$$D\mu[\varphi]=[\mathrm{d}\varphi]\int[\mathrm{d}\psi^*][\mathrm{d}\psi]\exp-\int L_F\,\mathrm{d}^Nx=[\mathrm{d}\varphi]\det\left[\frac{\delta\xi}{\delta\varphi}\right]=[\mathrm{d}\xi]$$

where the total Lagrangian is $L=L_B+L_F$.

For SUSYQM the Nicolai maps can be interpreted as stochastic processes [8]. In the above references [8], the formal properties of path integrals were used to show the equivalence of the Nicolai maps and SUSYQM. However it is well known that path integrals have ambiguities when defined at the discrete level, even in flat space, and in curved space the ambiguities are more numerous [9]. Since the Nicolai maps encountered in flat space SUSYQM are stochastic processes which can be defined at the discrete level [10], we intend to relate the discretisation of the stochastic process to the discretisation of the path integral by using the Nicolai map. Since there is a relationship between operator ordering of the Hamiltonian and the associated path integral, we should be able to find a relationship between the discretisation of the Nicolai map, the discretisation of the path integral, and operator ordering of the Hamiltonian.

First we shall start with SUSYQM on a flat manifold. The Hamiltonian is

$$H=\tfrac{1}{2}\eta^{ab}P_aP_b+\tfrac{1}{2}\eta^{ab}(\partial_aV)(\partial_bV)-\tfrac{1}{2}\eta^{ac}(\partial_a\partial_bV)[\psi^{*b},\psi_c].\tag{3.1}$$

The Nicolai map can be written as a stochastic process in the form

$$\Delta x_i^a=\eta^{ab}\{\partial_bV(\lambda_i)\}\Delta t_i+\Delta W_i^a\tag{3.2}$$

where $\Delta x_i^a=x_i^a-x_{i-1}^a$, $\lambda_i=x_{i-1}+\lambda\Delta x_i$, $\Delta W_i^a=w_i^a-w_{i-1}^a$, where w_i^a is a Wiener process [10]. This stochastic differential equation is just the integral of the Langevin equation $\dot{x}^a=\eta^{ab}(\partial_bV)+\xi^a$.

The probability density $P(w_i,t_i)$ satisfies

$$P(w_i,t_i)=\int\frac{d^Nw_{i-1}}{(2\pi\Delta t)^{N/2}}\exp\left(-\Delta t\frac{1}{2}\eta_{ab}\frac{\Delta W_i^a}{\Delta t_i}\frac{\Delta W_i^b}{\Delta t_i}\right)P(w_{i-1},t_{i-1}).\tag{3.3a}$$

Following [11], for infinitesimal time intervals Δt_i the above equation can be approximated by

$$P(w_i,t_i)=\int\frac{d^N(\Delta W_i)}{(2\pi\Delta t)^{N/2}}\exp\left(-\Delta t\frac{1}{2}\eta_{ab}\frac{\Delta W_i^a}{\Delta t_i}\frac{\Delta W_i^b}{\Delta t_i}\right)P(w_{i-1},t_{i-1})\tag{3.3b}$$

where it is understood that the path integral is for one infinitesimal time step. Using the stochastic differential equation (3.2), we perform a change of variables in (3.3*b*) from the stochastic variables (Wiener coordinates) to Cartesian coordinates, obtaining a path integral which corresponds to the no-fermion sector of SUSYQM,

$$\chi(x_i, t_i) = \int \frac{d^N(\Delta x)}{(2\pi\Delta t)^{N/2}} \det\left[\frac{\delta\Delta W}{\delta\Delta x}\right] \times\exp\left[-\Delta t\left\{\frac{1}{2}\eta_{ab}\frac{\Delta x^a}{\Delta t}\frac{\Delta x^b}{\Delta t}+\frac{1}{2}\eta^{ab}(\partial_a V)(\partial_b V) + \frac{1}{2}(1-2\lambda)\eta^{ab}(\partial_a\partial_b V)\right\}\right]\chi(x_{i-1}, t_{i-1}) \tag{3.4}$$

where $\chi(x_i, t_i) = P(x_i, t_i)\exp[-V(x_i)]$ and we have used the fact that [9]

$$\{\partial_a V(\lambda_i)\}\Delta x^a = V(x_i) - V(x_{i-1}) - \tfrac{1}{2}(1-2\lambda)\eta^{ab}\{\partial_a\partial_b V\}\Delta t_i. \tag{3.5}$$

Equation (3.5) is an important result in the manipulation of discrete path integrals. Formally we have the integral $\int_{t_0}^{t_f} dt(\partial_a V)\dot{x}^a = V(x_f) - V(x_0)$. But for the discrete path integral we use (3.5) to find that

$$\begin{aligned}\int_{t_0}^{t_f} dt(\partial_a V)\dot{x}^a &= \sum_{i=1}^{N}\{\partial_a V(\lambda_i)\}\Delta x_i^a \\ &= V(x_f) - V(x_0) - \tfrac{1}{2}(1-2\lambda)\sum_{i=1}^{N}\eta^{ab}\{\partial_a\partial_b V\}\Delta t_i\end{aligned}$$

which is not the naive result expected, but rather a λ dependent expression. The Jacobian for this change of variables is given by

$$\det\left[\frac{\delta\Delta W}{\delta\Delta x}\right] = \exp\left\{-\lambda\,\Delta t_i\eta^{ab}(\partial_a\partial_b V)\right\}. \tag{3.6}$$

Note that the λ dependent term in the integrand of (3.4) produced by (3.5) cancels the λ dependent term in the determinant, and hence the path integral is independent of λ.

In the spirit of part (*b*) of the definition of a Nicolai map, we can express equation (3.4) for the determinant in the form of a fermionic integral (see appendix 1 for our convention for the fermionic integrals):

$$\det\left[\frac{\delta\Delta W}{\delta\Delta x}\right] = \int\prod_{j=1}^{N}\{d\psi^{*j}(t_i)\,d\psi_j(t_i)\}\exp\left[\Delta\psi^{*a}(t_i)\psi_a(t_i)\right] \times\exp\left[\Delta t_i\eta^{ac}(\partial_a\partial_b V)\psi^{*b}(\lambda_i)\psi_c(t_i)\right] \tag{3.7}$$

where $\Delta\psi^{*a}(t_i) = \psi^{*a}(t_i) - \psi^{*a}(t_{i-1})$ and

$$\psi^{*b}(\lambda_i) = (1-\lambda)\psi^{*b}(t_i) + \lambda\psi^{*b}(t_{i-1}). \tag{3.8}$$

We see that equations (3.7) and (3.8) suggest an ordering for the fermionic variables in the definition of the path integral.

We can ask the question, will the above discretisation solve the ambiguities encountered in defining a discrete path integral for SUSYQM? Towards an answer to this question, we consider the SUSYQM of equation (3.1) but now described in curvilinear coordinates. We are assured that the Nicolai map exists, because we are still on a flat manifold.

The Hamiltonian in curvilinear coordinates is

$$H = \tfrac{1}{2}g^{-1/2}\tau_\mu g^{1/2}g^{\mu\nu}\tau_\nu + \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V) - \tfrac{1}{2}g^{\mu\tau}(\nabla_\mu\nabla_\nu V)[\psi^{*\nu}, \psi_\tau] \tag{3.9}$$

where $\tau_\mu = P_\mu + i\Gamma^\alpha_{\mu\beta}\psi^{*\beta}\psi_\alpha$.

In this case the stochastic process is given by

$$Dx^\mu(\lambda_i) = g^{\mu\nu}(\lambda_i)\{\partial_\nu V(\lambda_i)\}\Delta t_i + E^\mu_a(\lambda_i)\Delta W^a_i \tag{3.10}$$

which is the generalisation of equation (3.2) to curvilinear coordinates. E^μ_a are vielbeins with the property $\eta^{ab}E^\mu_a E^\nu_b = g^{\mu\nu}$, where μ, ν are curvilinear coordinate indices, and a, b are for Cartesian coordinate indices. The case $\lambda_i = 0$ corresponds to Itô calculus [10]. In the following calculation we use the fact that the manifold is flat, so that the vielbeins satisfy $\partial_\nu E^\mu_a + \Gamma^\mu_{\nu\lambda}E^\lambda_a = 0$. We can interpret

$$Dx^\mu(\lambda_i) = \Delta x^\mu + \tfrac{1}{2}(1-2\lambda)g^{\alpha\beta})(\lambda_i)\Gamma^\mu_{\alpha\beta}(\lambda_i)\Delta t_i \tag{3.11}$$

as the geodesic distance between x^μ_i and x^μ_{i-1}, where the geodesic is parametrised by λ_i such that $x^\mu(\lambda_i = 0) = x^\mu_{i-1}$ and $x^\mu(\lambda_i = 1) = x^\mu_i$. The velocity is then the geodesic distance divided by the time interval and we see that equation (2.10) is just the integral of the Langevin equation in curvilinear coordinate $\dot{x}^\mu = g^{\mu\nu}(\partial_\nu V) + \xi^\mu$.

Again we start with the Wiener process equation (3.3) and change variables to curvilinear coordinates, to give a path-integral which corresponds to the no-fermion sector of SUSYQM. Using the identity

$$\det[A+B] = \det A[1 + \mathrm{Tr}(1^{-1}B) + \tfrac{1}{2}[\mathrm{Tr}(A^{-1}B)]^2 - \tfrac{1}{2}\mathrm{Tr}(A^{-1}B)^2 + \ldots]$$

the determinant for this change of variables can be written as

$$\begin{aligned}\det\left[\frac{\delta\Delta W}{\delta\Delta x}\right] = g^{1/2}(\lambda_i)\bigg(&1 + \lambda\Gamma^\alpha_{\mu\alpha}(\lambda_i)\Delta x^\mu_i + \frac{\lambda}{2}(1-2\lambda)\{\partial_\mu g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\}\Delta t_i \\ &+ \frac{\lambda}{2}(1-2\lambda)g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\Gamma^\tau_{\mu\tau}\Delta t_i - \lambda g^{\mu\nu}\{\nabla_\mu\nabla_\nu V\}\Delta t_i \\ &+ \frac{\lambda^2}{2}g^{\mu\nu}\Gamma^\alpha_{\mu\alpha}\Gamma^\beta_{\nu\beta}\Delta t_i - \frac{\lambda^2}{2}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}\Delta t_i + \ldots\bigg)\end{aligned} \tag{3.12}$$

where $g(\lambda_i) = \det g^{\mu\nu}(\lambda_i)$, and

$$\begin{aligned}\chi(x_i, t_i) = \int \frac{d^N(\Delta x)}{(2\pi\Delta t)^{N/2}} J[x_i, x_{i-1}] \exp\bigg[-\Delta t\bigg\{&\tfrac{1}{2}g_{\mu\nu}(\lambda_i)\frac{Dx^\mu(\lambda_i)}{\Delta t}\frac{Dx^\nu(\lambda_i)}{\Delta t} \\ &+ \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V) + \tfrac{1}{2}(1-2\lambda)g^{\mu\nu}(\nabla_\mu\nabla_\nu V)\bigg\}\bigg]\chi(x_{i-1}, t_{i-1}).\end{aligned} \tag{3.13}$$

Here the Jacobian is

$$J[x_i, x_{i-1}] = g^{-1/2}(x_i)g^{1/2}(x_{i-1})\det\left[\frac{\delta\Delta W}{\delta\Delta x}\right]. \tag{3.14}$$

Now the scalar wave function is $\chi(x_i, t_i) = \exp[-V(x_i)]S(x_i, t_i)$, where we have the condition $\int \mathrm{d}x\, g^{1/2}S(x_i, t_i) = 1$. Previously we had the condition $\int \mathrm{d}x\, P(x_i, t_i) = 1$, hence $P(x_i, t_i) = g^{1/2}S(x_i, t_i)$ and this is the origin of the $g^{-1/2}(x_i)g^{1/2}(x_{i-1})$ term in the Jacobian which can be expressed as

$$g^{-1/2}(x_i)g^{1/2}(x_{i-1}) = 1 - \Gamma^\alpha_{\mu\alpha}(\lambda_i)\Delta x^\mu_i + \tfrac{1}{2}g^{\mu\nu}\Gamma^\alpha_{\mu\alpha}\Gamma^\beta_{\nu\beta}\Delta t_i - \tfrac{1}{2}(1-2\lambda)g^{\mu\nu}\{\partial_\mu\Gamma^\alpha_{\nu\alpha}\}\Delta t_i. \tag{3.15}$$

Equation (3.13) is just the point-canonical transformation of the path integral (3.4) and is a generalisation of the result of Gervais and Jevicki [6]. They showed that a naive change of variables of the path integral gives a different result than the change of variables of the discretised path integral. Here we see that the extra potential terms are contained in the Jacobian and in the geodesic distance used in the kinetic term. The result of (3.13) generalises that of Gervais and Jevicki to an arbitrary parameter λ in the interval $[0, 1]$, from the case they considered where $\lambda = \frac{1}{2}$.

There is a relationship between the path integral and operator ordering given by the following expressions. The one-time step kernel (where Δt is infinitesimal)

$$\frac{g^{1/2}(\lambda_i)}{(2\pi\Delta t)^{N/2}} \int \prod_{j=1}^{N} \{\mathrm{d}\psi^{*j}\,\mathrm{d}\psi_j\} \exp\left[-\Delta t\left\{\frac{1}{2} g_{\mu\nu}(\lambda_i) \frac{\Delta x^\mu}{\Delta t}\frac{\Delta x^\nu}{\Delta t} - \frac{\Delta\psi^{*\alpha}}{\Delta t}\psi_\alpha\right\}\right]$$
$$\times[1 - \alpha A_\mu(\lambda_i)\Delta x^\mu + \beta C\Delta t + \varepsilon\Gamma^\beta_{\mu\alpha}(\lambda_i)\Delta x^\mu \psi^{*\alpha}(\lambda_i)\psi_\beta$$
$$+ \gamma D_\alpha{}^\beta \psi^{*\alpha}(\lambda_i)\psi_\beta\Delta t + \delta E_\alpha{}^\beta{}_\gamma{}^\delta \psi^{*\alpha}(\lambda_i)\psi_\beta\psi^{*\gamma}(\lambda_i)\psi_\delta\Delta t] \qquad (3.16a)$$

leads to a Fokker-Planck equation (a Schrödinger equation with imaginary time) of the form

$$\partial_t\Psi = \tfrac{1}{2}(1-\lambda)\{\partial_\mu\partial_\nu g^{\mu\nu}\Psi\} + \tfrac{1}{2}\lambda g^{\mu\nu}\{\partial_\mu\partial_\nu\Psi\} - \tfrac{1}{2}\lambda(1-\lambda)\{\partial_\nu\partial_\nu g^{\mu\nu}\}\Psi$$
$$-\alpha[(1-\lambda)\{\partial_\nu g^{\mu\nu}A_\mu\Psi\} + \lambda g^{\mu\nu}A_\mu\{\partial_\nu\Psi\}]$$
$$-\varepsilon[(1-\lambda)\{\partial_\nu g^{\mu\nu}\Gamma^\beta_{\mu\alpha}[\psi^{*\alpha}\psi_\beta]_\lambda\Psi\} + \lambda g^{\mu\nu}\Gamma^\beta_{\mu\alpha}[\psi^{*\alpha}\psi_\beta]_\lambda\{\partial_\nu\Psi\}]$$
$$+\beta C\Psi + \gamma D_\alpha{}^\beta[\psi^{*\alpha}\psi_\beta]_\lambda\Psi + \gamma E_\alpha{}^\beta{}_\gamma{}^\delta[\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta]_\lambda\Psi, \qquad (3.16b)$$

where $\Psi = A_{ij\ldots n}(x)\psi^{*i}\ldots\psi^{*n}|0\rangle$ is a supersymmetric n-fermion state, ψ^{*i} and ψ_i are respectively the fermionic creation and annihilation operators that act on the fermionic vacuum $|0\rangle$. We have used the following notation: $[\psi^{*\mu}\psi_\nu]_\lambda = (1-\lambda)\psi^{*\mu}\psi_\nu - \lambda\psi_\nu\psi^{*\mu}$ and also $[\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta]_\lambda = [(1-\lambda)\psi^{*\alpha}\psi_\beta - \lambda\psi_\beta\psi^{*\alpha}][(1-\lambda)\psi^{*\gamma}\psi_\delta - \lambda\psi_\delta\psi^{*\gamma}]$.

Equations (3.16) are the main results of this paper: they give an explicit relationship between the path integral and the operator ordering for Hamiltonians which contain fermionic operators. We have not been able to find an analogous statement in the literature. However, DeAlfaro *et al* [12] speculated that the mid-point ($\lambda = \frac{1}{2}$) path integral leads to Hamiltonians which were Weyl ordered.

Starting with the Hamiltonian equation (3.9) and changing to imaginary time, we can use equation (3.16) to write the path integral as

$$\int \prod_{j=1}^{N} \frac{\{\mathrm{d}\psi^{*j}\,\mathrm{d}\psi_j\}}{(2\pi\Delta t)^{N/2}} T(x_i, x_{i-1})F[\psi^*(\lambda), \psi] \exp\left[-\Delta t\left\{\tfrac{1}{2}g_{\mu\nu}(\lambda_i)\frac{Dx^\mu(\lambda_i)}{\Delta t}\frac{Dx^\nu(\lambda_i)}{\Delta t}\right.\right.$$
$$\left.\left.+\tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V) + \tfrac{1}{2}(1-2\lambda)g^{\mu\nu}(\nabla_\mu\nabla_\nu V)\right\}\right] \qquad (3.17)$$

where the fermionic terms are contained in

$$F[\psi^*(\lambda), \psi]$$
$$= \exp[\Delta\psi^{*\alpha}\psi_\alpha]g^{1/2}(\lambda_i)[1 + \Gamma^\delta_{\mu\gamma}(\lambda_i)\psi^{*\gamma}(\lambda)\psi_\delta\Delta x^\mu$$
$$+(1-2\lambda)\Delta t g^{\mu\nu}\Gamma^\alpha_{\nu\alpha}\Gamma^\delta_{\mu\gamma}\psi^{*\gamma}\psi_\delta + \tfrac{1}{2}\Delta t\Gamma^\beta_{\mu\alpha}\Gamma^\delta_{\nu\gamma}\psi^{*\alpha}(\lambda)\psi_\beta\psi^{*\gamma}(\gamma)\psi_\delta$$
$$-(1-2\lambda)\Delta t g^{\mu\nu}\Gamma^\delta_{\nu\beta}\Gamma^\beta_{\mu\gamma}\psi^{*\gamma}(\lambda)\psi_\delta + \tfrac{1}{2}(1-2\lambda)\Delta t g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\Gamma^\delta_{\mu\gamma}\psi^{*\gamma}(\lambda)\psi_\delta$$
$$+\tfrac{1}{2}(1-2\lambda)\Delta t(\partial_\mu g^{\mu\nu}\Gamma^\delta_{\nu\gamma})\psi^{*\gamma}(\lambda)\psi_\delta + \Delta t g^{\mu\tau}(\nabla_\mu\nabla_\nu V)\psi^{*\nu}(\lambda)\psi_\tau]. \qquad (3.18)$$

We also have

$$T(x_i, x_{i-1}) = [1 - \tfrac{1}{2}(1-2\lambda)\Gamma^{\alpha}_{\mu\alpha}(\lambda)\Delta x^{\mu} - \tfrac{1}{4}(1-2\lambda)g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\tau}_{\mu\tau}\Delta t$$
$$+ \tfrac{1}{2}\{1-4\lambda(1-\lambda)\}(\partial_{\mu}g^{\mu\nu}\Gamma^{\beta}_{\nu\beta}\Delta t)]. \tag{3.19}$$

Upon comparing equations (3.13) and (3.17) we see that after integrating out the fermions in $F[\psi^*(\lambda), \psi]$ (see the appendix), the expression does not correspond to that of the Jacobian $J[x_i, x_{i-1}]$ for all values of the parameter λ. However for the special case $\lambda = \frac{1}{2}$, $T[x_i, x_{i-1}]|_{\lambda=\frac{1}{2}} = 1$ and $J[x_i, x_{i-1}]|_{\lambda=\frac{1}{2}} = F[\psi^*(\lambda), \psi]|_{\lambda=\frac{1}{2}}$. This suggests that the stochastic process of equation (3.10) with $\lambda = \frac{1}{2}$ is the Nicolai map for SUSYQM on a flat manifold in curvilinear coordinates. It should be noted that the path integrals in equations (3.13) and (3.17) give the correct Schrödinger equation for all possible values of the parameter λ. For $\lambda = \frac{1}{2}$ the stochastic process becomes the Nicolai map, hence $\lambda = \frac{1}{2}$ is the appropriate point about which to discretise. The existence of the Nicolai map in curvilinear coordinates resolves the ambiguity in the definition of the path integral by forcing us to choose $\lambda = \frac{1}{2}$.

Equations (3.4) and (3.13) are stochastic processes, hence the wavefunctions which are propagated by the kernels are just scalar wavefunctions. With the Jacobians replaced by the fermionic integrals (as in equation (3.20)), these kernels not only propagate scalar wavefunctions but also n-fermion states Ψ. By this we mean that we obtain the correct Schrödinger equation for the n-fermion states Ψ. The $\lambda = \frac{1}{2}$ parameter value corresponds to the mid-point rule in the path integral, and from equation (3.16) this is also equivalent to Weyl ordering of the operators in the Hamiltonian. This agrees with the results of [12] which were obtained in a different manner using the invariance of SUSYQM under general coordinate transformations to solve the operator ordering ambiguities in the Hamiltonian, and relating this to Weyl ordering.

The above results can now be used to quantise the SUSYQM given in the path integral form [13]. From equation (3.17) with $\lambda = \frac{1}{2}(=\bar{\lambda})$ and exponentiating all terms we obtain

$$\int \prod_{j=1}^{N} \{\mathrm{d}\psi * j\, \mathrm{d}\psi_j\} \frac{g^{1/2}(\lambda)}{(2\pi\Delta t)^{N/2}} \exp\Big[-\Delta t\Big\{\tfrac{1}{2}g_{\mu\nu}(\bar{\lambda})\frac{\Delta x^{\mu}}{\Delta t}\frac{\Delta x^{\nu}}{\Delta t} + \tfrac{1}{2}g^{\mu\nu}(\nabla_{\mu}V)(\nabla_{\nu}V)$$
$$-\frac{\Delta\psi^{*\alpha}}{\Delta t}\psi_{\alpha} - \Gamma^{\alpha}_{\mu\beta}(\bar{\lambda})\frac{\Delta x^{\mu}}{\Delta t}\psi^{*\beta}(\bar{\lambda})\psi_{\alpha} - g^{\alpha\tau}(\nabla_{\alpha}\nabla_{\beta}V)\psi^{*\beta}(\bar{\lambda})\psi_{\tau}\Big\}\Big] \tag{3.20}$$

which is the appropriate discretisation for the classical imaginary time SUSYQM action

$$S_E = \int \mathrm{d}\tau[\tfrac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \tfrac{1}{2}g^{\mu\nu}(\nabla_{\mu}V)(\nabla_{\nu}V) - \{\dot{\psi}^{*\alpha} + \Gamma^{\alpha}_{\mu\beta}\dot{x}^{\mu}\psi^{*\beta} + g^{\alpha\beta}(\nabla_{\tau}\nabla_{\beta}V)\psi^{*\tau}\}\psi_{\alpha}]. \tag{3.21}$$

4. Curved manifolds

In this section we explore the possibility of extending the Nicolai map to a curved Riemannian manifold. We assume that the Nicolai map for a curved manifold is again a stochastic process, and try to generalise the results of section 3 to curved manifolds. SUSYQM on a curved manifold is equivalent to the supersymmetric nonlinear σ-model in (0-1) dimensions, and is an example of supersymmetry that contains a non-trivial four-fermion term.

Using equation (2.7), equation (3.16) and the result of section 3, the path integral for the nonlinear σ-model at the mid-point is

$$\int \prod_{j=1}^{N} \{d\psi * j\, d\psi_j\} \frac{g^{1/2}(\bar\lambda)}{(2\pi\Delta t)^{N/2}} \exp\bigg[-\Delta t\bigg\{\tfrac{1}{2}g_{\mu bn}(\bar\lambda)\frac{\Delta x^\mu}{\Delta t}\frac{\Delta x^\nu}{\Delta t} + \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V) - \frac{\Delta\psi^{*\alpha}}{\Delta t}\psi_\alpha - \Gamma^\alpha_{\mu\beta}(\bar\lambda)\frac{\Delta x^\mu}{\Delta t}\psi^{*\beta}(\bar\lambda)\psi_\alpha - g^{\alpha\tau}(\nabla_\alpha\nabla_\beta V)\psi^{*\beta}(\bar\lambda)\psi_\tau + \tfrac{1}{2}R_\alpha{}^\beta{}_\gamma{}^\delta\psi^{*\alpha}(\bar\lambda)\psi_\beta\psi^{*\gamma}(\bar\lambda)\psi_\delta\bigg\}\bigg]. \tag{4.1}$$

Upon integrating the fermions we obtain

$$\frac{1}{(2\pi\Delta t)^{N/2}} V(\bar\lambda) \exp\bigg[-\Delta t\bigg\{\tfrac{1}{2}g_{\mu\nu}(\bar\lambda)\frac{\Delta x^\mu}{\Delta t}\frac{\Delta x^\nu}{\Delta} + \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V)\bigg\}\bigg] \tag{4.2}$$

where

$$V(\bar\lambda) = g^{1/2}(\bar\lambda)\exp\bigg[-\Delta t\bigg\{\tfrac{1}{2}\Gamma^\alpha_{\mu\alpha}(\bar\lambda)\frac{\Delta x^\mu}{\Delta t} + \tfrac{1}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \tfrac{1}{8}R - \tfrac{1}{2}g^{\mu\nu}\{\nabla_\mu\nabla_\nu V\}\bigg\}\bigg]. \tag{4.3}$$

We use vielbeins for the curved manifold, which relate natural coordinates (indices μ, ν) and orthonormal coordinates (indices a, b). The vielbeins satisfy

$$\begin{aligned}\partial_\tau E^\mu_a + \Gamma^\mu_{\tau\lambda}E^\lambda_a &= w^c{}_{a,\tau}E^\mu_c\\ \partial_\tau e^a_\mu - \Gamma^\lambda_{\tau\mu}e^a_\lambda &= -w^a{}_{c,\tau}e^c_\mu\end{aligned} \tag{4.4}$$

where e^a_μ is the inverse of E^μ_a, and $w^c{}_{a,\tau}$ is the spin connection.

First we shall consider the stochastic differential equation

$$Dx^\mu(\lambda_i) + \lambda\eta^{ab}E^\rho_b(\lambda_i)E^\mu_c(\lambda_i)w^c{}_{a,\rho}(\lambda_i)\Delta t_i = g^{\mu\nu}(\lambda_i)\{\partial_\nu V(\lambda_i)\}\Delta t_i + E^\mu_a(\lambda_i)\Delta W^a_i \tag{4.5}$$

as a possible candidate for the Nicolai map. We retrace the steps leading to equation (3.13), starting with equation (4.5) instead of equation (3.10) and working at the mid-point, as implied by the results of section 3. We find that

$$\chi(x_i, t_i) = \int \frac{d^N(\Delta x)}{(2\pi\Delta t)^{N/2}} J[x_i, x_{i-1}]\exp\bigg[-\Delta t\bigg\{\tfrac{1}{2}g_{\mu\nu}(\bar\lambda)\frac{\Delta x^\mu}{\Delta t}\frac{\Delta x^\nu}{\Delta t} + \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V) - \tfrac{1}{2}E^\rho_d(\bar\lambda)w^d{}_{f,\rho}(\bar\lambda)e^f_\nu(\bar\lambda)\frac{\Delta x^\nu}{\Delta t} - \tfrac{1}{2}\eta^{ed}E^\rho_d w^c{}_{e,\rho}E^\mu_c(\partial_\nu V) + \tfrac{1}{8}\eta^{ab}w^c{}_{a,\rho}E^\rho_c w^e{}_{b,\tau}E^\tau\bigg\}\bigg]\chi(x_{i-1}, t_{i-1}) \tag{4.6}$$

where the Jacobian $J[x_i, x_{i-1}]$ is given by equation (3.14)

$$J[x_i, x_{i-1}] = g^{1/2}(\bar\lambda)\exp\bigg[-\Delta t\bigg\{\tfrac{1}{2}\Gamma^\alpha_{\mu\alpha}(\bar\lambda)\frac{\Delta x^\mu}{\Delta t} + \tfrac{1}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} + \tfrac{1}{2}g^{\mu\nu}\{\nabla_\mu\nabla_\nu V\} - \tfrac{1}{2}E^\rho_d(\bar\lambda)w^d{}_{f,\rho}(\bar\lambda)e^f_\nu(\bar\lambda)\frac{\Delta x^\nu}{\Delta t} + \tfrac{1}{2}\eta^{ed}E^\rho_d w^c{}_{e,\rho}E^\mu_c(\partial_\nu V) - \tfrac{1}{4}\eta^{cd}\{\partial_\tau w^a{}_{c,\rho}\}E^\tau_a E^\rho_d - \tfrac{1}{8}\eta^{ab}w^c{}_{b,\tau}E^\tau_e w^e{}_{a,\rho}E^\rho_c\bigg\}\bigg]. \tag{4.7}$$

Using the identity $\frac{1}{4}\eta^{cd}\{\partial_\tau w^a{}_{c,\rho}\}E^\tau_a E^\rho_d + \frac{1}{8}\eta^{ab} w^c{}_{b,} E^\tau_e w^e{}_{a,\rho} E^\rho_c - \frac{1}{8}\eta^{ab} w^c{}_{a,\rho} E^\rho_c w^e{}_{b,\tau} E^\tau_e = \frac{1}{8}R$, we see that equation (4.6) is equivalent to equation (4.2). The stochastic differential equation (4.5) does in fact give the correct Fokker–Planck for the no-fermion state. Nevertheless, we note that this is not a Nicolai map as we have defined it above, because the Jacobian is not equal to the fermionic determinant of equation (4.3). Thus the above stochastic differential equation is appropriate for the propagation of scalar wavefunctions which satisfy the Fokker–Planck equation on a curved Riemannian manifold, however it is inappropriate as a Nicolai map for the no-fermion sector of SUSYQM on a curved manifold.

There is another way to construct stochastic differential equations which may be considered as candidates for the Nicolai map in curved space. We introduce auxiliary fields Φ^a_b which are orthogonal rotation matrices with inverse ϕ^c_d. The fields satisfy the following conditions

$$\begin{aligned} &\partial_\tau \Phi^a_b(\lambda_i) + w^a{}_{c,\tau}(\lambda_i)\Phi^c_b(\lambda_i) = 0 \\ &\partial_\tau \phi^a_b(\lambda_i) - \phi^a_c(\lambda_i) w^c{}_{b,\tau}(\lambda_i) = 0. \end{aligned} \tag{4.8}$$

If we now define transformed vielbeins by $\tilde{E}^\mu_a = E^\mu_b \Phi^b_a$ and $\tilde{e}^a_\mu = \phi^a_b e^b_\mu$, then for these transformed vielbeins we have

$$\begin{aligned} &\partial_\tau \tilde{E}^\mu_a(\lambda_i) = -\Gamma^\mu_{\tau\lambda}(\lambda_i)\tilde{E}^\lambda_a(\lambda_i) \\ &\partial_\tau \tilde{e}^a_\mu(\lambda_i) = \Gamma^\lambda_{\tau\mu}(\lambda_i)\tilde{e}^a_\lambda(\lambda_i). \end{aligned} \tag{4.9}$$

A straightforward generalisation of equation (3.10) leads us to consider the stochastic differential equation

$$Dx^\mu(\lambda_i) = g^{\mu\nu}(\lambda_i)\{\partial_\nu V(\lambda_i)\}\Delta t_i + \tilde{E}^\mu_a(\lambda_i)\Delta W^a_i \tag{4.10}$$

as the candidate for the Nicolai map. If we compare equations (4.9) and (4.10) with the corresponding equations in section 3, we see that the equations have the same structure. This implies that the above construction does not give the required Fokker–Planck equation on a curved manifold, but only generates a Fokker–Planck equation on a flat manifold.

This conclusion can be arrived at in another way. If $w^c{}_d = w^c{}_{c,\tau}\,dx^\tau$ is a connection one-form in a orthonormal frame and $\tilde{w}^a_b$ is another connection one-form in an overlapping orthonormal frame, then the two connection one-forms may be related by a gauge transformation $\tilde{w}^a_b = \phi^a_c[\mathrm{d}\Phi^c_b + w^c{}_d\Phi^d_b]$. The term in the bracket is zero by equation (4.8). A gauge transformation has been made to a frame in which the spin-connection is zero, this implies that the manifold is flat. In the paper by Claudson and Halpern [8] a similar construction was used; in our notation, this is given by $\dot{\Phi}^c_a + w^c{}_{b,\rho}\dot{x}^\rho\Phi^b_a = 0$. This is equivalent to $\mathrm{d}\Phi^c_a + w^c{}_b\Phi^b_a = 0$, which would imply that the gauge has been set such that the manifold is flat.

The above construction does not lead to a Nicolai map on a curved manifold, however it does indicate how we can construct one. We can augment equation (4.8) by including a term which is first order in Δx, then equation (4.8) becomes

$$\begin{aligned} &\partial_\tau \Phi^a_b(\lambda_i) + w^a{}_{c,\tau}(\lambda_i)\Phi^c_b(\lambda_i) = -\tfrac{1}{2}\lambda R^a{}_{c\tau\rho}(\lambda_i)\Delta x^\rho \Phi^c_b(\lambda_i) \\ &\partial_\tau \phi^a_b(\lambda_i) - \phi^a_c(\lambda_i) w^c{}_{b,\tau}(\lambda_i) = \tfrac{1}{2}\lambda \phi^a_c(\lambda_i) R^c{}_{b\tau\rho}(\lambda_i)\Delta x^\rho. \end{aligned} \tag{4.11}$$

Following the procedure of section 3 we obtain

$$\chi(x_i, t_i) = \int \frac{\mathrm{d}^N(\Delta x)}{(2\pi\Delta t)^{N/2}} J[x_i, x_{i-1}] \times \exp\left[-\Delta t\left\{\tfrac{1}{2}g_{\mu\nu}(\bar{\lambda})\frac{\Delta x^\mu}{\Delta t}\frac{\Delta x^\nu}{\Delta t} + \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu V)(\nabla_\nu V)\right\}\right]\chi(x_{i-1}, t_{i-1}) \tag{4.12}$$

where

$$J[x_i, x_{i-1}] = g^{1/2}(\bar{\lambda}) \times \exp\left[-\Delta t\left\{\tfrac{1}{2}\Gamma^\alpha_{\mu\alpha}(\bar{\lambda})\frac{\Delta x^\mu}{\Delta t} + \tfrac{1}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \tfrac{1}{8}R + \tfrac{1}{2}g^{\mu\nu}\{\nabla_\mu\nabla_\nu V\}\right\}\right]. \tag{4.13}$$

We see that the stochastic differential equation (4.10) along with equation (4.11) becomes the Nicolai map for a curved Riemannian manifold.

5. Conclusions

We have found that a careful treatment of the path integral for SUSYQM and for the associated stochastic process implies that at the midpoint the stochastic process can be identified as the Nicolai map, and the supersymmetric path integral contains no additional potential terms when compared to the formal path integral. This is evidence of the remarkable cancellation between the bosons and fermions in supersymmetry.

Although the path integral ambiguities were resolved by the explicit use of a Nicolai map in flat space, we note that the mid-point rule will also work on curved manifolds. By this we mean that the discrete path integral will contain no additional potential terms when compared to the formal path integral for the nonlinear σ-model on a curved manifold, and that the corresponding Hamiltonian will be Weyl ordered. Using the mid-point rule we were able to extend the Nicolai map to curved manifolds.

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Appendix

In this appendix we define our convention for fermionic integrals used in this paper.

$$\int \mathrm{d}\psi^{*\alpha}\,\psi^{*\beta} = \delta^{\alpha\beta} \qquad \int \mathrm{d}\psi^{*\alpha} = 0$$

$$\int \mathrm{d}\psi_\alpha\,\psi_\beta = \delta_{\alpha\beta} \qquad \int \mathrm{d}\psi_\alpha = 0.$$

Some useful integrals are

$$\int \prod_{j=1}^{N} \{d\psi^{*j}\, d\psi_j\} \exp(\Delta\psi^{*a}\psi_a) = 1$$

$$\int \prod_{j=1}^{N} \{d\psi^{*j}\, d\psi_j\} \exp(\Delta\psi^{*a}\psi_a)\psi^{*p} = \psi^{*p}(t_i)$$

$$\int \prod_{j=1}^{N} \{d\psi^{*j}\, d\psi_j\} \exp(\Delta\psi^{*a}\psi_a)\psi^{*a}(t_i)\psi_c\psi^{*p}) = \psi^{*a}(t_i)\delta_c^p$$

where $\Delta\psi^{*a} = \psi^{*a}(t_i) - \psi^{*a}(t_{i-1})$, $\psi^{*a} = \psi^{*a}(t_{i-1})$, and $\psi_a = \psi_a(t_{i-1})$.

The supersymmetric kernels propagate n-fermion states. As an example, consider the kernel in equation (3.20). We will propagate a one-fermion state $A_\sigma(x_{i-1}, t_{i-1})\psi^{*\sigma}|0\rangle$ to

$$A_\sigma(x_i, t_i)\psi^{*\sigma}(t_i)|0\rangle = \int dx_{i-1} K(x_i, t_i; x_{i-1}, t_{i-1}) A_\sigma(x_{i-1}, t_{i-1})\psi^{*\sigma}|0\rangle$$

where $K(x_i, t_i x_{i-1}, t_{i-1})$ is the kernel given in equation (3.20). Using the standard technique of expanding to order Δt and then doing the fermionic and bosonic integrals, we obtain the imaginary-time Schrödinger equation

$$\begin{aligned}(\partial_t A_\sigma)\psi^{*\sigma}(t_i)|0\rangle \\ &= [+\tfrac{1}{2}g^{\alpha\beta}(\nabla_\alpha\nabla_\beta A_\sigma) - g^{\alpha\beta}(\nabla_\sigma\nabla_\beta V)A_\alpha \\ &\quad + \tfrac{1}{2}g^{\alpha\beta}(\nabla_\alpha\nabla_\beta V)A_\sigma + \tfrac{1}{2}g^{\alpha\beta}(\nabla_\alpha V)(\nabla_\beta V)A_\sigma]\psi^{*\sigma}(t_i)|0\rangle.\end{aligned}$$

The generalisation to n-fermion states is straightforward.

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